Explaining maths

How to explain mathematical concepts in a way that students can understand

Sums and integration against counting measure: Part I

- August 24, 2023

For other posts in this series, see https://explaining-maths.blogspot.com/search?q=Sums+and+integration+against+counting+measure

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It has been a while since I wrote a mathematical post! Here I thought I would say something about Measure and Integration which is relevant to my research. For these posts I assume a basic knowledge of the Lebesgue theory of integration against measures, including the **Monotone Convergence Theorem**, **Fatou's Lemma** and the **Dominated Convergence Theorem**.

See the page https://explainingmaths.wordpress.com/measure-theory/ for links to my material on Measure Theory, and http://wp.me/posHB-da for links to a complete set of materials from a full module I gave on Measure and Integration in the years 2006-7-8.

You may also be interested in a paper I published in the Bulletin of the Irish Mathematical Society, Convergence from Below Suffices

It is well known that summing series is just the same as integrating against counting measure. To make this statement a bit more formal, let us take our domain X to be $\mathbb{N}$ (which excludes 0 in my teaching). Use the whole power set $\mathcal{P}(\mathbb{N})$ as our sigma algebra, and take our measure μ to be counting measure on $\mathcal{P}(\mathbb{N})$. We will work with values in the non-negative extended reals, $[0, \infty]$. In this setting, every function f from $X = \mathbb{N}$ to $[0, \infty]$ is measurable, and we have

$$\int_X f \, \mathrm{d}\mu = \sum_{n=1}^{\infty} f(n) \, .$$

More generally, let X be **any** set, take our sigma algebra to be $\mathcal{P}(X)$, and take μ to be counting measure on $\mathcal{P}(X)$. Let $f : X \to [0, \infty]$. Then f is measurable, and we can **define**

$$\sum_{x \in X} f(x) = \int_X f \, \mathrm{d}\mu \, .$$

This agrees with our usual notions of summation over finite sets or countably infinite sets, but is also valid when X is uncountably infinite.

It can also be useful to note that, in this setting,

$$\sum_{x \in X} f(x) = \sup \left\{ \sum_{x \in E} f(x) : E \text{ is a finite subset of } X \right\} .$$

This isn't very hard to prove, but I may say something about it in a later post. It doesn't look to be quite immediate from the definitions as I originally thought.

In the next few posts I will look at how to interpret the main theorems of Lebesgue integration theory in the setting of sums (including series). In the process we will recover many well-known standard facts about series of non-negative real numbers (value of the sum does not depend on the order you add the terms, changing order of integration in double sums, etc.). We will also look at the convergence theorems, and note that these can be strengthened on the special case of counting measure (and other suitable measures). For that discussion, we will assume some knowledge of **nets** (generalised sequences, indexed by directed sets). We recall that the three main convergence theorems of Lebesgue integration theory mentioned earlier are results about **sequences** of functions, but that these results are not generally true for **nets** of functions. However, if we are working with **counting measure**, those theorems **do** become true for nets of functions too. This doesn't appear to be very well known (though I have found some discussion of it on the web). For my own work, I am particularly interested in nets indexed by initial intervals of ordinals, which I think of as "ordinal sequences". (Is that a standard term?)

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