Explaining maths

How to explain mathematical concepts in a way that students can understand

Sums and integration against counting measure: Part II

- August 24, 2023

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In this post we will recover some familiar results about summing non-negative real numbers. In fact we will work with non-negative extended real numbers, and see how to interpret the usual results in terms of integration against counting measure.

Let X be a countably infinite set (for example $\mathbb{N}$), let μ be counting measure on X (or more formally on $\mathcal{P}(X)$), and let $f : X \to [0, \infty]$. In our previous post, we said that we could **define**

$$\sum_{x \in X} f(x) = \int_X f \, \mathrm{d}\mu \,.$$

An alternative is to consider a bijection $g : \mathbb{N} \to X$ (an enumeration of X), and attempt to define

$$\sum_{x \in X} f(x) = \sum_{n=1}^{\infty} f(g(n)) \,.$$

However, if we do it this way, we face the problem of showing that the result is independent of which bijection g we chose. **In other words, we need to show that the order does not matter when we add up these non-negative extended real numbers.** This is, essentially, the well-known fact that if you add a series with **non-negative terms**, you always get the same result even if you rearrange the order of the terms. (Indeed, in the case where $X = \mathbb{N}$ that is almost exactly what this says.)

You don't need measure theory to prove any of this. However, it does maybe provide a helpful way to look at it. Because the integral against counting measure above clearly doesn't depend on g at all. So all we need to prove is that, whatever bijection $g : \mathbb{N} \to X$ we choose, we always get

$$\sum_{n=1}^{\infty} f(g(n)) = \int_X f\,\mathrm{d}\mu\,,$$

and then we will be in good shape!

So, with X and f as above, let g be a bijection from $\mathbb{N}$ to X. Then X is a disjoint union of a sequence of single-point sets

$$X = \bigcup_{n=1}^{\infty} \{g(n)\}\,.$$

Then (using any suitable standard Measure and Integration results), we have

$$\int_X f\,\mathrm{d}\mu = \sum_{n=1}^{\infty} \int_{\{g(n)\}} f\,\mathrm{d}\mu = \sum_{n=1}^{\infty} f(g(n))\,,$$

as required.

So we do always get the same answer from any such bijection g, and the answer is given by the integral of f against counting measure, as claimed.

There are some other standard results for series with non-negative terms (rearranging, regrouping, changing the order of double sums, etc.). These are also quite easy to explain in terms of integrating against counting measure. I'll look at that in the next post.

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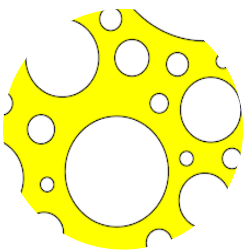

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